Construction of Volatility Surface for Commodity Futures

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Market Implied Volatility

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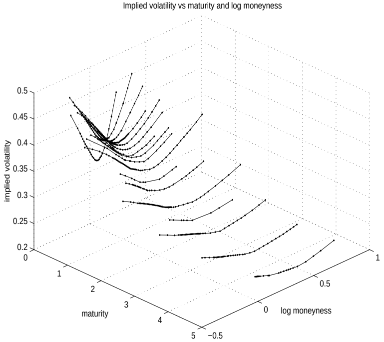

Figure: Implied volatilities from crude oil market

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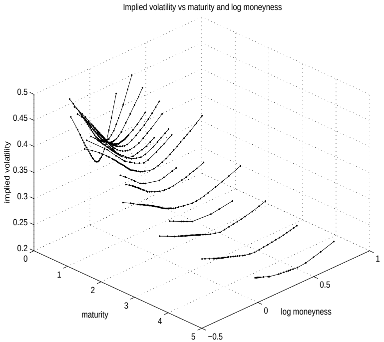

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- How to incorporate this info into pricing and risk?

Model Overview

- Extract info from the implied volatility surface
- Commodity market: two key effects in the surface:
  - Samuelson effect (term structure of ATM volatility)
  - Volatility Smiles (marginal distributions of the underlying futures)

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- Our approach is to calibrate the two effects separately:
  - A volatility model to calibrate the term structure
  - Local volatility model to interpolate the smiles
- A market modeling approach:
  - Direct modeling of the forward prices (market observable)
  - Use copula to recover the join distribution for pricing and risk

Volatility Term Structure Calibration

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- Log-normal framework:

$$dF(t, T_i) = \sigma(t, T_i)F(t, T_i)dW_i(t), \quad t \leq T_i, \tag{1}$$

for $i = 1, \cdots, n$

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- The instantaneous volatility $\sigma(t, T)$ is a function of time t
- Calibration: specify a volatility model $\sigma(t, T)$

Log-Normal Model: Examples

- Schwartz-Smith model (2000):
  - Volatility

$$\sigma^2(t,T) = \sigma_X^2 e^{-2\kappa(T-t)} + 2\rho_{XY}\sigma_X\sigma_Y e^{-\kappa(T-t)} + \sigma_Y^2$$

  - Correlation

$$dW(t) = \sigma_X \frac{e^{-\kappa(T-t)}}{\sigma(t,T)} dW_X(t) + \sigma_Y \frac{1}{\sigma(t,T)} dW_Y(t)$$

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- Gabillon model (1991):
  - Volatility

$$\sigma^2(t,T) = \sigma_S^2 e^{-2\kappa(T-t)} + 2\rho_{SL}\sigma_S\sigma_L \left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right) + \sigma_L^2 \left(1 - e^{-\kappa(T-t)}\right)^2$$

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Calibrated Results

- The model-implied volatility:

$$\sigma_{mod}(\tau, T) = \sqrt{\frac{1}{\tau}\int_0^{\tau} \sigma^2(t, T)dt}, \quad \tau \leq T \tag{2}$$

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- Gabillon model: $\kappa = 0.37$, $\rho_{SL} = -0.29$, $\sigma_S = 0.41$, $\sigma_L = 0.29$

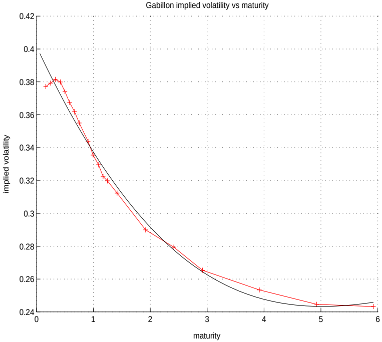

Figure: Volatility term structure (ATM)

Market Approach: Exogenous Correlation

- Forward model: flexibility to model correlation and volatility
- A correlation model given by Ronn (2009):

$$\rho_{ij} = e^{-b|T_i - T_j|} + (1 - e^{-b|T_i - T_j|})e^{-a/\min(T_i,T_j)}, \quad (a, b > 0) \tag{3}$$

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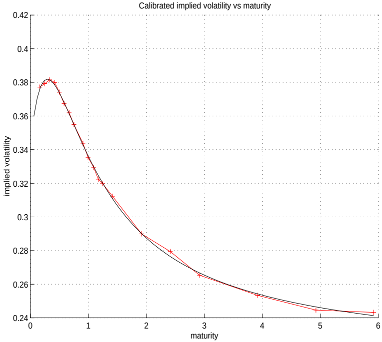
- Much better fit: $a = 1.10$, $b = 3.38$, $b = 0.13$, $d = 0.21$



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Distribution Surface

- Marginal distribution can be calculated from call price:

$$\psi(\tau, K) := \mathbb{P}(\tilde{F}(\tau, T) < K) = 1 + \frac{\partial}{\partial K} C(\tau, K) \quad (5)$$

- How to construct the surfaces $C(t, K; \tau, T)$ for $t < \tau \leq T$?
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  - Interpolate the resulted curves in the time dimension
  - Potential issues: accuracy, stability and arbitrage
- New alternative: local volatility model
  - Apply the Dupire equation to perform the interpolation
  - Andreasen-Huge (2011): local volatility surface in FX market

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Option Prices on Discrete Expiries

- A set of forward maturities: $0 = T_0 < T_1 < \cdots < T_n$
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- Now fix a forward maturity T_i:
- Given local volatility $\vartheta_{ij}(x) := \sigma_{loc}(\tau_j, x; T_i)$ constant in $t \in [\tau_{j-1}, \tau_j]$
- Construct the call prices for all option expiries by solving:

$$\left[1 - \frac{1}{2}\Delta\tau_j \vartheta_{ij}^2(x)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\right] C(\tau_j, x; T_i) = C(\tau_{j-1}, x; T_i) \tag{9}$$

  with $C(0, x; T_i) = F(0, T_i)(1 - e^x)^+$, $(1 \leq j \leq i \leq n)$
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- Define a volatility smile for option expiries τ_j by scaling:

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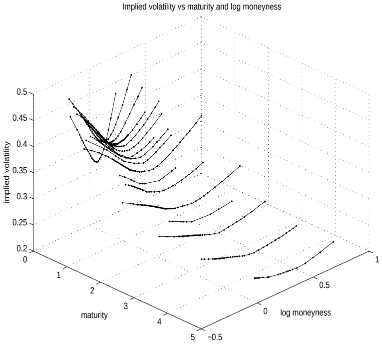
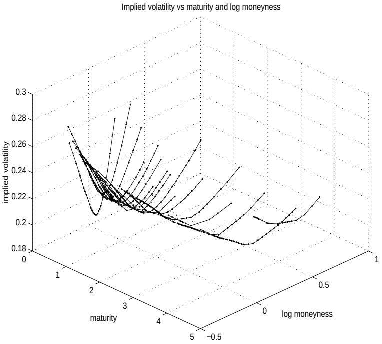
Smile Scaling: Example

- Consider the forward with maturity $T = 4.93$



(a) Market Volatility



(b) Scaled Volatility

- Remove most of the Samuelson effect

Calibration $\vartheta_{ij}(x)$

- Discretize the local volatility function $\vartheta_{ij}(x)$:
  - For a fixed maturity-expiry pair (i,j), let $\vartheta_{ij}(x_{ijk}) = \theta_{ijk}$
  - Define the function $\vartheta_{ij}(x)$ through interpolation:

$$\vartheta_{ij}(x) = h(\{x_{ijk}, \theta_{ijk}\}_k)$$

- Solve the optimization problem:

$$\min_{\Theta_{ij}} \sum_k \left[C(\tau_j, x_{ijk}; T_i, \theta_{ijk}) - \hat{C}(\tau_j, x_{ijk}; T_i)\right]^2 \qquad (12)$$

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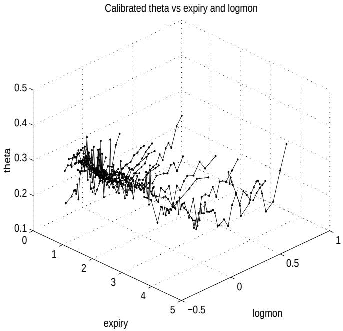
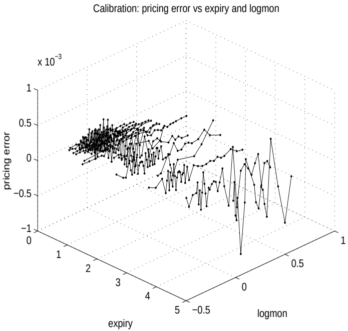
Calibration results

- Calibrated $\vartheta_{ij}(x)$ and pricing errors (maturity $T = 4.93$)



(c) Calibrated theta



(d) Pricing errors

- Underlying forward price: \$89.15
- The RMSE of call price: \$3.3e-4 (5-year options)

Fill the Gaps between Options Expiries

- Use the calibrated local volatility to interpolate the intermediate option prices at $t \in (\tau_{j-1}, \tau_j)$
- Again, by solving the PDE

$$\left[1 - \frac{1}{2}(t - \tau_{j-1})\vartheta_{ij}^2(x)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\right] C(t, x; T_i) = C(\tau_{j-1}, x; T_i),$$

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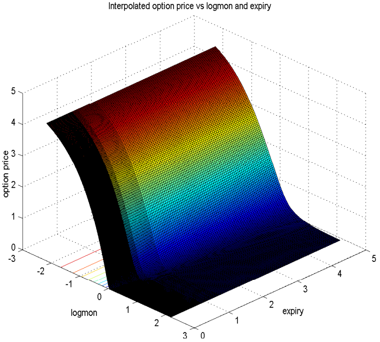
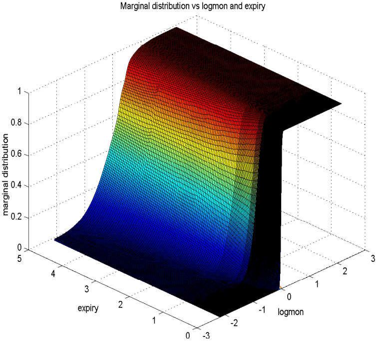
Price Surface ⇒ Distribution Surface

- Marginal distribution surface: $\psi(\tau, K) = 1 + \frac{\partial}{\partial K} C(\tau, K)$



(e) Interpolated call option prices

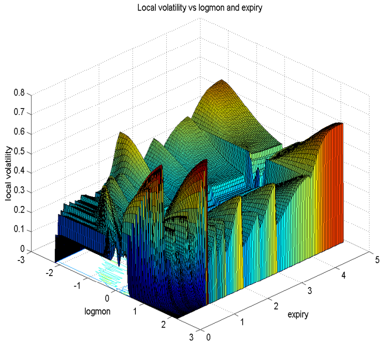


(f) Marginal distribution surface

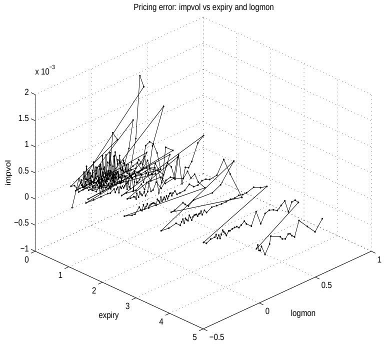
- One surface for each forward contract (maturity $T = 4.93$)

Local Volatility Surface

- Local volatility surface and pricing errors (maturity $T = 4.93$)



(g) Local volatility surface



(h) Pricing errors

- The RMSE of ImpVol = 1.6 bps

Pricing Errors

- Re-price the options using the imaginary local volatility surfaces
- Calculate the implied volatilities and pricing errors (vs market)
- The resulted RMSE:

Expiry (y)	0.08	0.16	0.25	0.33	0.42	0.50
RMSE (bp)	2.0	2.9	2.6	2.2	1.8	1.8
Expiry (y)	0.58	0.67	0.75	0.91	1.17	1.42
RMSE (bp)	1.7	1.4	1.8	1.5	1.6	2.3
Expiry (y)	1.92	2.41	2.92	3.92	4.93	
RMSE (bp)	2.2	2.0	2.4	2.6	1.6	

Table: The RMSE of implied volatility

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- Calculate the implied volatilities and pricing errors (vs market)
- The resulted RMSE:

Expiry (y)	0.08	0.16	0.25	0.33	0.42	0.50
RMSE (bp)	2.0	2.9	2.6	2.2	1.8	1.8
Expiry (y)	0.58	0.67	0.75	0.91	1.17	1.42
RMSE (bp)	1.7	1.4	1.8	1.5	1.6	2.3
Expiry (y)	1.92	2.41	2.92	3.92	4.93	
RMSE (bp)	2.2	2.0	2.4	2.6	1.6	

Table: The RMSE of implied volatility

Log-normal Distribution

- How to obtain the joint distribution for pricing and risk?
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where

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Gaussian Copula

- The normal variable $Z_i(t)$ can be transformed to a uniform:

$$U_i(t) = \Phi(Z_i(t)), \quad (i = 1, \cdots, n)$$

where $\Phi(\cdot)$ is a normal CDF

- The joint distribution of the uniforms define a copula function:

$$c(u_1, \cdots, u_n) := F_{U_1(t), \cdots, U_n(t)}(u_1, \cdots, u_n)$$

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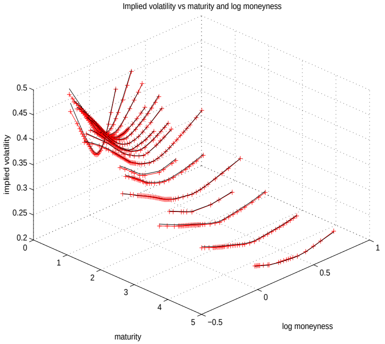
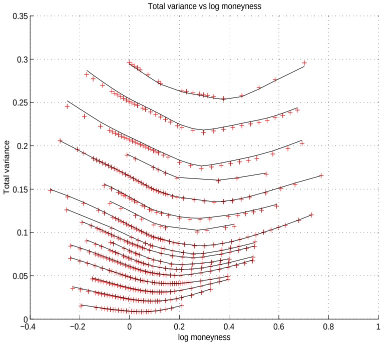
Simulation Results

- Re-price the options using MC simulation
- Calculate implied volatility (total variance) vs market quotes



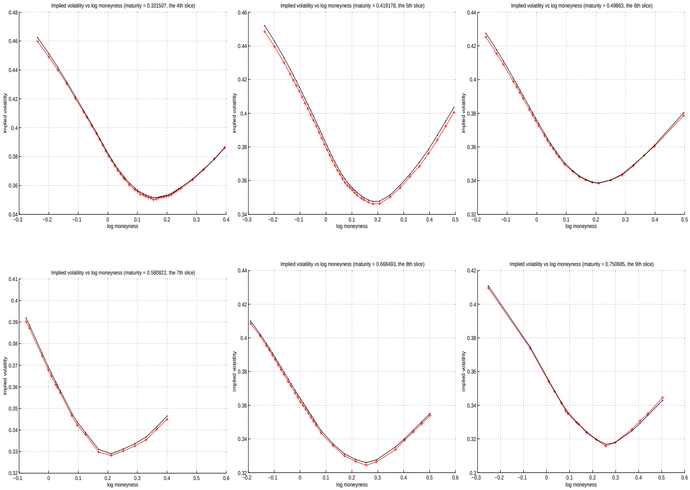
(i) Implied volatility



(j) Total variance

Simulation Results, cont.

- Individual smiles: market vs model (from MC simulation)



Thank you !

Technical Reference

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- Qimou Su and Curt Randall, Construction of Volatility Surface for Commodity Futures, Working paper, 2012